Supplement of "Improved Dropout for Shallow and Deep Learning"

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1 Proof of Theorem 1

The update given by $\mathbf{w}_{t+1} = \mathbf{w}_t - \eta\nabla\ell(\mathbf{w}_t^\top(\mathbf{x}_t \circ \boldsymbol{\epsilon}_t), y_t)$ can be considered as the stochastic gradient descent (SGD) update of the following problem

$$\min_{\mathbf{w}} \{\widehat{\mathcal{L}}(\mathbf{w}) \triangleq \mathrm{E}_{\widehat{\mathcal{P}}}[\ell(\mathbf{w}^\top(\mathbf{x} \circ \boldsymbol{\epsilon}), y)]\}$$

Define $\mathbf{g}_t$ as $\mathbf{g}_t = \nabla\ell(\mathbf{w}_t^\top(\mathbf{x}_t \circ \boldsymbol{\epsilon}_t), y_t) = \ell'(\mathbf{w}_t^\top(\mathbf{x}_t \circ \boldsymbol{\epsilon}_t), y_t)\mathbf{x}_t \circ \boldsymbol{\epsilon}_t$, where $\ell'(z, y)$ denotes the derivative in terms of z. Since the loss function is G-Lipschitz continuous, therefore $\|\mathbf{g}_t\|_2 \leq G\|\mathbf{x}_t \circ \boldsymbol{\epsilon}_t\|_2$. According to the analysis of SGD [3], we have the following lemma.

Lemma 1. *Let $\mathbf{w}_{t+1} = \mathbf{w}_t - \eta\mathbf{g}_t$ and $\mathbf{w}_1 = 0$. Then for any $\|\mathbf{w}_*\|_2 \leq r$ we have*

$$\sum_{t=1}^n \mathbf{g}_t^\top(\mathbf{w}_t - \mathbf{w}_*) \leq \frac{r^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^n \|\mathbf{g}_t\|_2^2 \tag{1}$$

By taking expectation on both sides over the randomness in $(\mathbf{x}_t, y_t, \boldsymbol{\epsilon}_t)$ and noting the bound on $\|\mathbf{g}_t\|_2$, we have

$$\mathrm{E}_{[n]}\left[\sum_{t=1}^n \mathbf{g}_t^\top(\mathbf{w}_t - \mathbf{w}_*)\right] \leq \frac{r^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^n G^2\mathrm{E}_{[n]}[\|\mathbf{x}_t \circ \boldsymbol{\epsilon}_t\|_2^2]$$

where $\mathrm{E}_{[t]}$ denote the expectation over $(\mathbf{x}_i, y_i, \boldsymbol{\epsilon}_i), i = 1, \ldots, t$. Let $\mathrm{E}_t[\cdot]$ denote the expectation over $(\mathbf{x}_t, y_t, \boldsymbol{\epsilon}_t)$ with $(\mathbf{x}_i, y_i, \boldsymbol{\epsilon}_i), i = 1, \ldots, t-1$ given. Then we have

$$\sum_{t=1}^n \mathrm{E}_{[t]}[\mathbf{g}_t^\top(\mathbf{w}_t - \mathbf{w}_*)] \leq \frac{r^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^n G^2\mathrm{E}_t[\|\mathbf{x}_t \circ \boldsymbol{\epsilon}_t\|_2^2]$$

Since

$$\mathrm{E}_{[t]}[\mathbf{g}_t^\top(\mathbf{w}_t - \mathbf{w}_*)] = \mathrm{E}_{[t-1]}[\mathrm{E}_t[\mathbf{g}_t]^\top(\mathbf{w}_t - \mathbf{w}_*)] = \mathrm{E}_{[t-1]}[\nabla\widehat{\mathcal{L}}(\mathbf{w}_t)^\top(\mathbf{w}_t - \mathbf{w}_*)] \geq \mathrm{E}_{[t-1]}[\widehat{\mathcal{L}}(\mathbf{w}_t) - \widehat{\mathcal{L}}(\mathbf{w}_*)]$$

As a result

$$\mathrm{E}_{[n]}\left[\sum_{t=1}^n (\widehat{\mathcal{L}}(\mathbf{w}_t) - \widehat{\mathcal{L}}(\mathbf{w}_*))\right] \leq \frac{r^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^n G^2\mathrm{E}_{\widehat{\mathcal{D}}}[\|\mathbf{x}_t \circ \boldsymbol{\epsilon}_t\|_2^2] \leq \frac{r^2}{2\eta} + \frac{\eta}{2}G^2B^2n \tag{2}$$

where the last inequality follows the assumed upper bound of $\mathrm{E}_{\widehat{\mathcal{D}}}[\|\mathbf{x}_t \circ \boldsymbol{\epsilon}_t\|_2^2]$. Following the definition of $\widehat{\mathbf{w}}_n$ and the convexity of $\mathcal{L}(\mathbf{w})$ we have

$$\mathrm{E}_{[n]}[\widehat{\mathcal{L}}(\widehat{\mathbf{w}}_n) - \widehat{\mathcal{L}}(\mathbf{w}_*)] \leq \mathrm{E}_{[n]}\left[\frac{1}{n}\sum_{t=1}^n (\widehat{\mathcal{L}}(\mathbf{w}_t) - \widehat{\mathcal{L}}(\mathbf{w}_*))\right] \leq \frac{r^2}{2\eta n} + \frac{\eta}{2}G^2B^2$$

30th Conference on Neural Information Processing Systems (NIPS 2016), Barcelona, Spain.

By minimizing the upper bound in terms of η, we have $\mathrm{E}_{[n]}[\widehat{\mathcal{L}}(\widehat{\mathbf{w}}_n) - \widehat{\mathcal{L}}(\mathbf{w}_*)] \leq \frac{GBr}{\sqrt{n}}$. According to Proposition 1 in the paper $\widehat{\mathcal{L}}(\mathbf{w}) = \mathcal{L}(\mathbf{w}) + R_{\mathcal{D},\mathcal{M}}(\mathbf{w})$, therefore

$$\mathrm{E}_{[n]}[\mathcal{L}(\widehat{\mathbf{w}}_n) + R_{\mathcal{D},\mathcal{M}}(\widehat{\mathbf{w}}_n)] \leq \mathcal{L}(\mathbf{w}_*) + R_{\mathcal{D},\mathcal{M}}(\mathbf{w}_*) + \frac{GBr}{\sqrt{n}}$$

1.1 Proof of Lemma 1

We have the following:

$$\frac{1}{2}\|\mathbf{w}_{t+1} - \mathbf{w}_*\|_2^2 = \frac{1}{2}\|\mathbf{w}_t - \eta\mathbf{g}_t - \mathbf{w}_*\|_2^2 = \frac{1}{2}\|\mathbf{w}_t - \mathbf{w}_*\|_2^2 + \frac{\eta^2}{2}\|\mathbf{g}_t\|_2^2 - \eta(\mathbf{w}_t - \mathbf{w}_*)^\top\mathbf{g}_t$$

Then

$$(\mathbf{w}_t - \mathbf{w}_*)^\top\mathbf{g}_t \leq \frac{1}{2\eta}\|\mathbf{w}_t - \mathbf{w}_*\|_2^2 - \frac{1}{2\eta}\|\mathbf{w}_{t+1} - \mathbf{w}_*\|_2^2 + \frac{\eta}{2}\|\mathbf{g}_t\|_2^2$$

By summing the above inequality over $t = 1, \ldots, n$, we obtain

$$\sum_{t=1}^{n}\mathbf{g}_t^\top(\mathbf{w}_t - \mathbf{w}_*) \leq \frac{\|\mathbf{w}_* - \mathbf{w}_1\|_2^2}{2\eta} + \frac{\eta}{2}\sum_{t=1}^{n}\|\mathbf{g}_t\|_2^2$$

By noting that $\mathbf{w}_1 = 0$ and $\|\mathbf{w}_*\|_2 \leq r$, we obtain the inequality in Lemma 1.

2 Proof of Proposition 2

We have

$$\mathrm{E}_{\widehat{\mathcal{D}}}\|\mathbf{x} \circ \boldsymbol{\epsilon}\|_2^2 = \mathrm{E}_{\mathcal{D}}\left[\sum_{i=1}^{d}\frac{x_i^2}{k^2p_i^2}\mathrm{E}[m_i^2]\right]$$

Since $\{m_1, \ldots, m_d\}$ follows a multinomial distribution $Mult(p_1, \ldots, p_d; k)$, we have

$$\mathrm{E}[m_i^2] = var(m_i) + (\mathrm{E}[m_i])^2 = kp_i(1 - p_i) + k^2p_i^2$$

The result in the Proposition follows by combining the above two equations.

3 Proof of Proposition 3

Note that only the first term in the R.H.S of Eqn. (7) depends on p_i. Thus,

$$\mathbf{p}_* = \arg\min_{\mathbf{p}\geq 0, \mathbf{p}^\top\mathbf{1}=1}\sum_{i=1}^{d}\frac{\mathrm{E}_{\mathcal{D}}[x_i^2]}{p_i}$$

The result then follows the KKT conditions.

4 Proof of Proposition 4

We prove the first upper bound first. From Eqn. (4) in the paper, we have

$$\widehat{R}_{\mathcal{D},\mathcal{M}}(\mathbf{w}_*) \leq \frac{1}{8}\mathrm{E}_{\mathcal{D}}[\mathbf{w}_*^\top C_{\mathcal{M}}(\mathbf{x} \circ \epsilon)\mathbf{w}_*]$$

where we use the fact $\sqrt{ab} \leq \frac{a+b}{2}$ for $a, b \geq 0$. Using Eqn. (5) in the paper, we have

$$\mathrm{E}_{\mathcal{D}}[\mathbf{w}_*^\top C_{\mathcal{M}}(\mathbf{x} \circ \epsilon)\mathbf{w}_*] = \mathrm{E}_{\mathcal{D}}\left[\mathbf{w}_*^\top\left(\frac{1}{k}diag(x_i^2/p_i) - \frac{1}{k}\mathbf{x}\mathbf{x}^\top\right)\mathbf{w}_*\right] = \frac{1}{k}\mathrm{E}_{\mathcal{D}}\left[\sum_{i=1}^{d}\frac{w_{*i}^2x_i^2}{p_i} - (\mathbf{w}_*^\top\mathbf{x})^2\right]$$

This gives a tight bound of $\widehat{R}_{\mathcal{D},\mathcal{M}}(\mathbf{w}_*)$, i.e.,

$$\widehat{R}_{\mathcal{D},\mathcal{M}}(\mathbf{w}_*) \leq \frac{1}{8k}\left\{\sum_{i=1}^{d}\frac{w_{*i}^2\mathrm{E}_{\mathcal{D}}[\mathbf{x}_i^2]}{p_i} - \mathrm{E}_{\mathcal{D}}(\mathbf{w}_*^\top\mathbf{x})^2\right\}$$

By minimizing the above upper bound over p_i, we obtain following probabilities

$$p_i^* = \frac{\sqrt{w_{*i}^2 \mathrm{E}_{\mathcal{D}}[x_i^2]}}{\sum_{j=1}^d \sqrt{w_{*i}^2 \mathrm{E}_{\mathcal{D}}[x_j^2]}} \tag{3}$$

which depend on unknown $\mathbf{w}_*$. We address this issue, we derive a relaxed upper bound. We note that

$$\begin{aligned} C_{\mathcal{M}}(\mathbf{x} \circ \epsilon) &= \mathrm{E}_{\mathcal{M}}[(\mathbf{x} \circ \boldsymbol{\epsilon} - \mathbf{x})(\mathbf{x} \circ \boldsymbol{\epsilon} - \mathbf{x})^\top] \\ &\leq (\mathrm{E}_{\mathcal{M}}\|\mathbf{x} \circ \boldsymbol{\epsilon} - \mathbf{x}\|_2^2) \cdot I_d = \left(\mathrm{E}_{\mathcal{M}}[\|\mathbf{x} \circ \boldsymbol{\epsilon}\|_2^2] - \|\mathbf{x}\|_2^2\right) I_d \end{aligned}$$

where I_d denotes the identity matrix of dimension d. Thus

$$\mathrm{E}_{\mathcal{D}}[\mathbf{w}_*^\top C_{\mathcal{M}}(\mathbf{x} \circ \epsilon)\mathbf{w}_*] \leq \|\mathbf{w}_*\|_2^2 \left(\mathrm{E}_{\widehat{\mathcal{D}}}[\|\mathbf{x} \circ \boldsymbol{\epsilon}\|_2^2] - \mathrm{E}_{\mathcal{D}}[\|\mathbf{x}\|_2^2]\right)$$

By noting the result in Proposition 2 in the paper, we have

$$\mathrm{E}_{\mathcal{D}}[\mathbf{w}_*^\top C_{\mathcal{M}}(\mathbf{x} \circ \epsilon)\mathbf{w}_*] \leq \frac{1}{k}\|\mathbf{w}_*\|_2^2 \left(\sum_{i=1}^d \frac{\mathrm{E}_{\mathcal{D}}[\mathbf{x}_i^2]}{p_i} - \mathrm{E}_{\mathcal{D}}[\|\mathbf{x}\|_2^2]\right)$$

which proves the upper bound in Proposition 4.

5 Neural Network Structures

In this section we present the neural network structures and the number of filters, filter size, padding and stride parameters for MNIST, SVHN, CIFAR-10 and CIFAR-100, respectively. Note that in Table 2, Table 3 and Table 4, the rnorm layer is the local response normalization layer and the local layer is the locally-connected layer with unshared weights.

5.1 MNIST

We used the similar neural network structure to [2]: two convolution layers, two fully connected layers, a softmax layer and a cost layer at the end. The dropout is added to the first fully connected layer. Tables 1 presents the neural network structures and the number of filters, filter size, padding and stride parameters for MNIST.

Table 1: The Neural Network Structure for MNIST

Layer Type	Input Size	#Filters	Filter size	Padding/Stride	Output Size
conv1	$28 \times 28 \times 1$	32	4×4	0/1	$21 \times 21 \times 32$
pool1(max)	$21 \times 21 \times 32$		2×2	0/2	$11 \times 11 \times 32$
conv2	$11 \times 11 \times 32$	64	5×5	0/1	$7 \times 7 \times 64$
pool2(max)	$7 \times 7 \times 64$		3×3	0/3	$3 \times 3 \times 64$
fc1	$3 \times 3 \times 64$				150
dropout	150				150
fc2	150				10
softmax	10				10
cost	10				1

5.2 SVHN

The neural network structure used for this data set is from [2], including 2 convolutional layers, 2 max pooling layers, 2 local response layers, 2 fully connected layers, a softmax layer and a cost layer with one dropout layer. Tables 2 presents the neural network structures and the number of filters, filter size, padding and stride parameters used for SVHN data set.

5.3 CIFAR-10

The neural network structure is adopted from [2], which consists two convolutional layer, two pooling layers, two local normalization response layers, 2 locally connected layers, two fully connected layers and a softmax and a cost layer. Table 3 presents the detail neural network structure and the number of filters, filter size, padding and stride parameters used.

Table 2: The Neural Network Structure for SVHN

Layer Type	Input Size	#Filters	Filter Size	Padding/Stride	Output Size
conv1	$28 \times 28 \times 3$	64	5×5	0/1	$24 \times 24 \times 64$
pool1(max)	$24 \times 24 \times 64$		3×3	0/2	$12 \times 12 \times 64$
rnorm1	$12 \times 12 \times 64$				$12 \times 12 \times 64$
conv2	$12 \times 12 \times 64$	64	5×5	2/1	$12 \times 12 \times 64$
rnorm2	$12 \times 12 \times 64$				$12 \times 12 \times 64$
pool2(max)	$12 \times 12 \times 64$		3×3	0/2	$6 \times 6 \times 64$
local3	$6 \times 6 \times 64$	64	3×3	1/1	$6 \times 6 \times 64$
local4	$6 \times 6 \times 64$	32	3×3	1/1	$6 \times 6 \times 32$
dropout	1152				1152
fc1	1152				512
fc10	512				10
softmax	10				10
cost	10				1

Table 3: The Neural Network Structure for CIFAR-10

Layer Type	Input Size	#Filters	Filter Size	Padding/Stride	Output Size
conv1	$24 \times 24 \times 3$	64	5×5	2/1	$24 \times 24 \times 64$
pool1(max)	$24 \times 24 \times 64$		3×3	0/2	$12 \times 12 \times 64$
rnorm1	$12 \times 12 \times 64$				$12 \times 12 \times 64$
conv2	$12 \times 12 \times 64$	64	5×5	2/1	$12 \times 12 \times 64$
rnorm2	$12 \times 12 \times 64$				$12 \times 12 \times 64$
pool2(max)	$12 \times 12 \times 64$		3×3	0/2	$6 \times 6 \times 64$
local3	$6 \times 6 \times 64$	64	3×3	1/1	$6 \times 6 \times 64$
local4	$6 \times 6 \times 64$	32	3×3	1/1	$6 \times 6 \times 32$
dropout	1152				1152
fc1	1152				128
fc10	128				10
softmax	10				10
cost	10				1

5.4 CIFAR-100

The network structure for this data set is similar to the neural network structure in [1], which consists of 2 convolution layers, 2 max pooling layers, 2 local response normalization layers, 2 locally connected layers, 3 fully connected layers, and a softmax and a cost layer. Table 4 presents the neural network structures and the number of filters, filter size, padding and stride parameters used for CIFAR-100 data set.

5.5 The Neural Network Structure used for BN

Tables 5 and 6 present the network structures of different methods in subsection 5.3 in the paper. The layer pool(ave) in Table 5 and Table 6 represents the average pooling layer.

References

[1] Alex Krizhevsky and Geoffrey Hinton. Learning multiple layers of features from tiny images, 2009.

[2] Li Wan, Matthew Zeiler, Sixin Zhang, Yann L Cun, and Rob Fergus. Regularization of neural networks using dropconnect. In *Proceedings of the 30th International Conference on Machine Learning (ICML-13)*, pages 1058–1066, 2013.

Table 4: The Neural Network Structure for CIFAR-100

Layer Type	Input Size	#Filters	Filter Size	Padding/Stride	Output Size
conv1	$32 \times 32 \times 3$	64	5×5	2/1	$32 \times 32 \times 64$
pool1(max)	$32 \times 32 \times 64$		3×3	0/2	$16 \times 16 \times 64$
rnorm1	$16 \times 16 \times 64$				$16 \times 16 \times 64$
conv2	$16 \times 16 \times 64$	64	5×5	2/1	$16 \times 16 \times 64$
rnorm2	$16 \times 16 \times 64$				$16 \times 16 \times 64$
pool2(max)	$16 \times 16 \times 64$		3×3	0/2	$8 \times 8 \times 64$
local3	$8 \times 8 \times 64$	64	3×3	1/1	$8 \times 8 \times 64$
local4	$8 \times 8 \times 64$	32	3×3	1/1	$8 \times 8 \times 32$
fc1	2048				128
dropout	128				128
fc2	128				128
fc100	128				100
softmax	100				100
cost	100				1

Table 5: Layers of networks for the experiment comparing with BN on CIFAR-10

Layer Type	noBN-noDropout	BN	e-dropout
Layer 1	conv1	conv1	conv1
Layer 2	pool1(max)	pool(max)	pool1(max)
Layer 3	N/A	bn1	N/A
Layer 4	conv2	conv2	conv2
Layer 5	N/A	bn2	N/A
Layer 6	pool2(ave)	pool2(ave)	pool2(ave)
Layer 7	conv3	conv3	conv3
Layer 8	N/A	bn3	e-dropout
Layer 9	pool3(ave)	pool3(ave)	pool3(ave)
Layer 10	fc1	fc1	fc1
Layer 11	softmax	softmax	softmax

Table 6: Sizes in networks for the experiment comparing with BN on CIFAR-10

Layer Type	Input size	#Filters	Filter size	Padding/Stride	Output size
conv1	$32 \times 32 \times 3$	32	5×5	2/1	$32 \times 32 \times 32$
pool1(max)	$32 \times 32 \times 32$		3×3	0/2	$16 \times 16 \times 32$
conv2	$16 \times 16 \times 32$	32	5×5	2/1	$16 \times 16 \times 32$
pool2(ave)	$16 \times 16 \times 32$		3×3	0/2	$8 \times 8 \times 32$
conv3	$8 \times 8 \times 32$	64	5×5	2/1	$8 \times 8 \times 64$
pool3(ave)	$8 \times 8 \times 64$		3×3	0/2	$4 \times 4 \times 64$
fc1	$4 \times 4 \times 64$				10
softmax	10				10
cost	10				1

[3] Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the International Conference on Machine Learning (ICML)*, pages 928–936, 2003.